

Tensor powers for non-simply laced Lie algebras B_2 -case

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Abstract

We study the decomposition problem for tensor powers of B_2 -fundamental modules. To solve this problem singular weight technique and injection fan algorithms are applied. Properties of multiplicity coefficients are formulated in terms of multiplicity functions. These functions are constructed showing explicitly the dependence of multiplicity coefficients on the highest weight coordinates and the tensor power parameter. It is thus possible to study general properties of multiplicity coefficients for powers of the fundamental B_2 - modules.

1 Introduction

Consider an analog of the Brauer centralizer algebras for the spinor groups and define the subspaces of the tensor space $(\otimes^p \mathbf{V}^n)$ on which the symmetric group S_k and $\text{Spin}(n)$ act as the dual pair (in a direct product form). Here \mathbf{V}^n is the fundamental representation of $\text{Spin}(n)$. Namely the centralizer algebra of the orthogonal group in $(\otimes^p \mathbf{V}^n)$ is generated by the symmetric group S_k and the contractions and the immersions of the invariant form and is called the Brauer centralizer algebras. To proceed further one needs the list of $\text{Spin}(n)$ -irreducible subspaces in the decomposition $(\otimes^p \mathbf{V}^n) = \sum_{\mu \in P} m_{\mu}^p \mathbf{V}^{(\mu)}$ (P is the $\text{Spin}(n)$ weight space) and their multiplicities m_{μ}^p . As far as we are interested in an arbitrary power p of the fundamental module $\mathbf{V}_{\text{Spin}(n)}^n$ our main problem is to find multiplicities of submodules in a form of *multiplicity functions*

$M(\mu, p)$ explicitly depending on the corresponding highest weight μ and the power parameter p .

There are numerous combinatorial studies of the problem [1, 2, 3, 4] and also series of works dealing with fermionic formulas, some of them based on crystal basis approach [5, 6, 7, 8]. On this way important general results were obtained [9, 10, 11]. On the other hand practical computations with the corresponding formulas are scarcely possible for all but the simplest examples. In most of these studies the simply laced algebras are considered and as a rule the multiplicities formulas are connected with complicated path countings.

We must mention also an algorithm for tensor product decompositions proposed in [12] and improved in [13]. It is used in our investigations.

Summing up, we are to find multiplicities m_μ^p in the decomposition $(\otimes^p \mathbf{V}^n) = \sum_{\mu \in P} m_\mu^p \mathbf{V}^{(\mu)}$ as a function of μ and p . To solve this problem we propose an algorithm based on singular weights properties [14] and the injection fan technique [15, 16]. We study the multiplicities m_μ^p formulated in terms of multiplicity functions $M_{\mathfrak{g}}(\mu, p)$. The latter have the weight space \mathcal{LP} for the domain of definition. On the sublattice P^{++} of dominant weights the multiplicity function gives us the desired multiplicities, $M(\mu, p)|_{\mu \in P^{++}} = m_\mu^p$. In this paper we shall show how to adopt these tools to non-simply laced algebras and shall demonstrate how they work by studying the tensor powers $(L_{B_n}^{\omega_i})^{\otimes p}$ of the fundamental module $L_{B_2}^{\omega_i}$ of B_2 .

2 Basic definitions and relations.

\mathfrak{g} – simple Lie algebra of the series B_n ,

L^μ – the integrable module of \mathfrak{g} with the highest weight μ ;

r – the rank of the algebra \mathfrak{g} ;

Δ – the root system; Δ^+ – the positive root system for \mathfrak{g} ;

\mathcal{N}^μ – the weight diagram of L^μ ;

W – the Weyl group;

$C^{(0)}$ – the fundamental Weyl chamber, $\overline{C^{(0)}}$ – its closure;

ρ – the Weyl vector;

$\epsilon(w) := \det(w)$, $w \in W$;

α_i – the i -th simple root for \mathfrak{g} ; $i = 0, \dots, r$;

ω_i – the i -th fundamental weight for \mathfrak{g} ; $i = 0, \dots, r$;

$L_{\mathfrak{g}}^{\omega_i}$ – the i -th fundamental module;

$\{e^i\}_{i=1, \dots, r}$ – the natural Euclidean basis of the weight space (the e -basis),

$\{v_i\}$ – the coordinates of a weight in the e -basis;

P – the weight lattice \mathcal{LP} – the weight space;

Q – the root lattice;

\mathcal{E} – the group algebra of the group P ;

$\Psi^{(\mu)} := \sum_{w \in W} \epsilon(w) e^{w \circ (\mu + \rho) - \rho}$ – the singular element for the \mathfrak{g} -module L^μ ;

$\widehat{\Psi^{(\mu)}}$ – the set of singular weights $\psi \in P$ for the module L^μ with the coordinates $(\psi, \epsilon(w(\psi))) \mid \psi = w(\psi) \circ (\mu + \rho) - \rho$;
 $\text{ch}(L^\mu)$ – the formal character of L^μ ;
 $\text{ch}(L^\mu) = \frac{\sum_{w \in W} \epsilon(w) e^{w \circ (\mu + \rho) - \rho}}{\prod_{\alpha \in \Delta^+} (1 - e^{-\alpha})} = \frac{\Psi^{(\mu)}}{\Psi^{(0)}}$ – the Weyl formula;
 $R := \prod_{\alpha \in \Delta^+} (1 - e^{-\alpha}) = \Psi^{(0)}$ – the denominator;
 $M_{\mathfrak{g}}^{\omega_i}(\mu, p)$ – the multiplicity function corresponding to the decomposition $(L_{\mathfrak{g}}^{\omega_i})^{\otimes p} = \sum m_{\mu}^{(i)p} L_{\mathfrak{g}}^{\mu}$, $M_{\mathfrak{g}}^{\omega_i}(\mu, p) \mid_{\mu \in P^{++}} = m_{\mu}^{(i)p}$.

3 Some useful properties.

Lemma 1 *The projection $\Psi_{\downarrow \mathfrak{g}}^{(\nu, \xi)}$ of the singular element $\Psi_{\mathfrak{g} \oplus \mathfrak{g}}^{(\nu, \xi)}$ for the irreducible representation $L^{(\nu_1, \dots, \nu_r)} \otimes L^{(\xi_1, \dots, \xi_r)}$ of the direct sum $\mathfrak{g} \oplus \mathfrak{g}$ on the weight space of the diagonal subalgebra $\mathfrak{g} \rightarrow \mathfrak{g} \oplus \mathfrak{g}$. is equal to the product*

$$\Psi_{\downarrow \mathfrak{g}}^{(\nu, \xi)} = \Psi_{\mathfrak{g}}^{\nu} \Psi_{\mathfrak{g}}^{\xi},$$

Let $\left\{ \psi_k^{(\nu)} \mid \psi_k^{(\nu)} \in \Psi^{(\nu)}, k = 1, \dots, \#W \right\}$ and $\left\{ \psi_p^{(\xi)} \mid \psi_p^{(\xi)} \in \Psi^{(\xi)}, p = 1, \dots, \#W \right\}$ be the sets of singular weights for the modules $L^{(\nu_1, \dots, \nu_r)}$ and $L^{(\xi_1, \dots, \xi_r)}$ correspondingly then the set $\widehat{\Psi_{\downarrow \mathfrak{g}}^{(\nu, \xi)}}$ consists of the weights $\left\{ \psi_k^{(\nu)} + \psi_p^{(\xi)}, \epsilon \left(w \left(\psi_k^{(\nu)} \right) \right) \epsilon \left(w \left(\psi_p^{(\xi)} \right) \right) \right\}$.

Proof. Let $\{e^1, e^2, \dots, e^r, e^{r+1}, \dots, e^{2r}\}$ be the weight space basis for $\mathcal{LP}(\mathfrak{g} \oplus \mathfrak{g})$, $(\nu_1, \dots, \nu_r, \xi_1, \dots, \xi_r)$ – the coordinates for the highest weight (ν, ξ) naturally belonging to the space $\mathcal{LP}(\mathfrak{g} \oplus \mathfrak{g})$. The weights $v^{(\nu)} \in \mathcal{N}^{\nu}$, $u^{(\xi)} \in \mathcal{N}^{\xi}$ and the singular vectors $\psi_k^{(\nu)}$ and $\psi_p^{(\xi)}$ also are lifted to the space $\mathcal{LP}(\mathfrak{g} \oplus \mathfrak{g})$

$$\begin{aligned}
lv_a^{(\nu)} &\Rightarrow (\nu_{a1}^{(\nu)}, \dots, \nu_{ar}^{(\nu)}, 0, \dots, 0); lu_b^{(\xi)} \Rightarrow (0, \dots, 0, u_{b1}^{(\xi)}, \dots, u_{br}^{(\xi)}) \\
l\psi_k^{(\nu)} &\Rightarrow (\psi_{k1}^{(\nu)}, \dots, \psi_{kr}^{(\nu)}, 0, \dots, 0); l\psi_p^{(\xi)} \Rightarrow (0, \dots, 0, \psi_{p1}^{(\xi)}, \dots, \psi_{pr}^{(\xi)})
\end{aligned}$$

The set $\{lv_a + lu_b \mid a = 1, \dots, \dim(L^\nu), b = 1, \dots, \dim(L^\xi)\}$ forms the weight diagram $\mathcal{N}_{\mathfrak{g} \oplus \mathfrak{g}}^{(\nu, \xi)}$ of $L_{\mathfrak{g} \oplus \mathfrak{g}}^{(\nu, \xi)}$. As far as for the Weyl group $W_{\mathfrak{g} \oplus \mathfrak{g}}$ we have $W_{\mathfrak{g} \oplus \mathfrak{g}} = W \times W$ and the Weyl vector is $\rho_{\mathfrak{g} \oplus \mathfrak{g}} = (\rho, \rho)$, the set of singular weights $\widehat{\Psi_{\mathfrak{g} \oplus \mathfrak{g}}^{(\nu, \xi)}}$ is formed by the vectors whose first $2r$ coordinates are $\{l\psi_k^{(\nu)} + l\psi_p^{(\xi)} \mid k, p = 1, \dots, \#W\}$ and the last one is equal to the product $\epsilon \left(w \left(\psi_k^{(\nu)} \right) \right) \epsilon \left(w \left(\psi_p^{(\xi)} \right) \right)$:

$$\widehat{\Psi_{\mathfrak{g} \oplus \mathfrak{g}}^{(\nu, \xi)}} = \left\{ l\psi_k^{(\nu)} + l\psi_p^{(\xi)}, \epsilon \left(w \left(\psi_k^{(\nu)} \right) \right) \epsilon \left(w \left(\psi_p^{(\xi)} \right) \right) \mid k, p = 1, \dots, \#W \right\}.$$

The vector $(c_1, \dots, c_r, c_{r+1}, \dots, c_{2r}) \in \mathcal{LP}(\mathfrak{g} \oplus \mathfrak{g})$ being projected to the diagonal subalgebra weight space \mathcal{LP} in the basis $\{(e^1 + e^{r+1})/2, \dots, (e^r + e^{2r})/2\}$

has the coordinates $((c_1 + c_{r+1}), \dots, (c_r + c_{2r}))$. The latter means that

$$\begin{aligned} \widehat{\Psi}_{\downarrow \mathfrak{g}}^{(\nu, \xi)} &= \left\{ \psi_k^{(\nu)} + \psi_p^{(\xi)}, \epsilon \left(w \left(\psi_k^{(\nu)} \right) \right) \epsilon \left(w \left(\psi_p^{(\xi)} \right) \right) \right\}, \\ k, p &= 1, \dots, \#W. \end{aligned}$$

■

Q.E.D.

One of the main tools to study the decomposition properties is the *injection fan* $\Gamma_{\mathfrak{g}^{\text{diag}} \rightarrow \mathfrak{g} \oplus \mathfrak{g}}$. [15, 16]. To use this instrument we consider the decomposition of tensor products as a special case of branching. The latter corresponds to the injection of the diagonal subalgebra into the direct sum: $\mathfrak{g}^{\text{diag}} \rightarrow \mathfrak{g} \oplus \mathfrak{g}$.

Lemma 2 *The vectors of the injection fan Γ for $\mathfrak{g}^{\text{diag}} \rightarrow \mathfrak{g} \oplus \mathfrak{g}$ consists of the opposites to the singular weights of the trivial module $L_{\mathfrak{g}}^{(0)}$*

$$\Gamma_{\mathfrak{g}^{\text{diag}} \rightarrow \mathfrak{g} \oplus \mathfrak{g}} = -S \circ \widehat{\Psi}^{(0)} \setminus (0, \dots, 0),$$

(here S is the full reflection).

Proof. According to the definition [15] the vectors γ of the fan $\Gamma_{\mathfrak{g}^{\text{diag}} \rightarrow \mathfrak{g} \oplus \mathfrak{g}}$ are fixed by the relation $1 - \prod_{\alpha \in (\Delta_{\mathfrak{g} \oplus \mathfrak{g}}^+)} (1 - e^{-\alpha})^{\text{mult}_{\mathfrak{g} \oplus \mathfrak{g}}(\alpha) - \text{mult}_{\mathfrak{g}^{\text{diag}}}(\alpha)} = \sum_{\gamma \in \Gamma_{\mathfrak{g}^{\text{diag}} \rightarrow \mathfrak{g} \oplus \mathfrak{g}}} s(\gamma) e^{-\gamma}$. The projections of the $\mathfrak{g} \oplus \mathfrak{g}$ -roots to the diagonal subalgebra obviously reproduce the set $\{\alpha_i\}_{i=0, \dots, r}$ in $\mathcal{LP}(\mathfrak{g})$:

$$\left. \begin{aligned} (\alpha_i, 0)_{\downarrow \mathfrak{g}} \\ (0, \alpha_i)_{\downarrow \mathfrak{g}} \end{aligned} \right\} = \alpha_i.$$

Thus $\text{mult}_{\mathfrak{g}}(\alpha) = 2$ while $\text{mult}_{\mathfrak{g} \oplus \mathfrak{g}}(\alpha) = 1$ and we have

$$\begin{aligned} \sum_{\gamma \in \Gamma_{\mathfrak{g}^{\text{diag}} \rightarrow \mathfrak{g} \oplus \mathfrak{g}}} s(\gamma) e^{-\gamma} &= 1 - \prod_{\alpha \in (\Delta_{\mathfrak{g} \oplus \mathfrak{g}}^+)} (1 - e^{-\alpha})^{\text{mult}_{\mathfrak{g} \oplus \mathfrak{g}}(\alpha) - \text{mult}_{\mathfrak{g}}(\alpha)} = \\ &= 1 - \prod_{\alpha \in (\Delta^+)} (1 - e^{-\alpha}). \end{aligned}$$

Q.E.D. ■

Lemma 3 *The singular element $\Psi_{\mathfrak{g}}^{(\xi)}$ for the module $L^{\mu} \otimes L^{\nu}$ can be presented in two equivalent forms:*

$$\text{ch}(L_{\mathfrak{g}}^{\mu}) \Psi_{\mathfrak{g}}^{\nu} = \Psi_{\mathfrak{g}}^{\mu} \text{ch}(L_{\mathfrak{g}}^{\nu}). \quad (1)$$

Proof. In the Weyl formula for $L^{\mu} \otimes L^{\nu}$,

$$\text{ch}(L^{\mu} \otimes L^{\nu})_{\downarrow P_{\mathfrak{g}}^+} = \sum_{\xi \in P_{\mathfrak{g}}^{++}} m_{\xi}^{\mu\nu} \text{ch}(L^{\xi}),$$

apply the result of Lemma 1:

$$\left(\frac{\Psi_{\mathfrak{g} \oplus \mathfrak{g}}^{(\nu, \xi)}}{\Psi_{\mathfrak{g} \oplus \mathfrak{g}}^0} \right)_{\downarrow P_{\mathfrak{g}}} = \frac{\Psi_{\mathfrak{g}}^{\mu} \Psi_{\mathfrak{g}}^{\nu}}{\Psi_{\mathfrak{g}}^0 \Psi_{\mathfrak{g}}^0} = \sum_{\xi \in P_{\mathfrak{g}}^{++}} m_{\xi}^{\mu\nu} \frac{\Psi_{\mathfrak{g}}^{\xi}}{\Psi_{\mathfrak{g}}^0}.$$

$$\left((\Psi_{\mathfrak{g}}^0)^{-1} \Psi_{\mathfrak{g}}^{\mu} \right) \Psi_{\mathfrak{g}}^{\nu} = \Psi_{\mathfrak{g}}^{\mu} \left((\Psi_{\mathfrak{g}}^0)^{-1} \Psi_{\mathfrak{g}}^{\nu} \right) = \sum_{\xi \in P_{\mathfrak{g}}^{++}} m_{\xi}^{\mu\nu} \Psi_{\mathfrak{g}}^{\xi}.$$

Thus we have

$$\sum_{\xi \in P_{\mathfrak{g}}^{++}} m_{\xi}^{\mu\nu} \Psi_{\mathfrak{g}}^{\xi} = \text{ch} (L_{\mathfrak{g}}^{\mu}) \Psi_{\mathfrak{g}}^{\nu} = \Psi_{\mathfrak{g}}^{\mu} \text{ch} (L_{\mathfrak{g}}^{\nu}). \quad (2)$$

Q.E.D. ■

Now put $\mu = \omega$, $\nu = (p-1)\omega$

$$\text{ch} (L^{(\omega)}) \Psi^{(\otimes^{(p-1)} \omega)} = \sum_{\xi \in P} M_{\mathfrak{g}}^{\omega_1} (\xi, p) \Psi^{(\xi)}, \quad (3)$$

$M_{\mathfrak{g}}^{\omega_1} (\xi, p)$ defines the singular element $\Psi^{(\otimes^p \omega)}$. On the other hand these equations can be considered as a system of recurrent relations for the multiplicity function $M_{\mathfrak{g}}^{\omega_1} (\xi, p)$.

Conjecture 4 Let $\mathfrak{g} = B_2$, L^{μ} and $L^{\omega_1} = L^{vect}$ be the highest weight modules with the highest weights μ and ω_1 (the first fundamental weight). Then the tensor product decomposition $(L^{\mu} \otimes L^{vect}) \downarrow_{\mathfrak{g}_{\text{diag}}} = \oplus_{\gamma} L^{\gamma}$ is multiplicity free.

Proof. According to Lemma 1 the projected singular element for a $\mathfrak{g} \oplus \mathfrak{g}$ -module $L^{\mu} \otimes L^{vect}$ is $\Psi_{\mathfrak{g} \oplus \mathfrak{g} \downarrow \mathfrak{g}_{\text{diag}}}^{(\mu, vect)} = \Psi_{\mathfrak{g}}^{\mu} \Psi_{\mathfrak{g}}^{vect}$ and the set of singular weights is

$$\widehat{\Psi_{\downarrow \mathfrak{g}_{\text{diag}}}^{(\mu, vect)}} = \left\{ \psi_k^{(\mu)} + \psi_p^{(vect)}, \epsilon \left(w \left(\psi_k^{(\mu)} \right) \right) \epsilon \left(w \left(\psi_p^{(vect)} \right) \right) \right\},$$

$$k, p = 1, \dots, \#W.$$

Suppose $\mu = (\mu_1, \mu_2)$ is greater than $\omega_1 = (1, 0)$ and $\mu_1 > \mu_2 \geq 1$. The singular weights of L^{μ} in the fundamental chamber and its nearest neighbours are

$$\left\{ \psi_s^{(\mu)} \right\} = (\mu_1, \mu_2, (+1)), (\mu_1, -\mu_2 - 1, (-1)), (\mu_2 - 1, \mu_1 + 1, (-1))$$

$$s = 1, 2, 3$$

They will give rise to the 24 weights of the type

$$\left\{ \psi_s^{(\mu)} + \psi_p^{(vect)} \right\} = (\mu_1, \mu_2, (+1)) + \psi_p^{(vect)},$$

$$(\mu_1, -\mu_2 - 1, (-1)) + \psi_p^{(vect)},$$

$$(\mu_2 - 1, \mu_1 + 1, (-1)) + \psi_p^{(vect)}$$

$$s = 1, 2, 3; p = 1, \dots, \#W.$$

Applying successively the fan $\Gamma_{\mathfrak{g}^{\text{diag}} \rightarrow \mathfrak{g} \oplus \mathfrak{g}}$ to the set $\widehat{\Psi_{\downarrow \mathfrak{g}^{\text{diag}}}^{(\mu, \text{vect})}}$ in the three selected chambers (starting with the highest weight $(\mu_1 + 1, \mu_2, (+1))$) we find the weights:

$$\begin{aligned} &(\mu_1, \mu_2, (+1)), (\mu_1 \pm 1, \mu_2, (+1)), (\mu_1, \mu_2 \pm 1, (+1)), \\ &(\mu_1, -\mu_2 - 1, (-1)), (\mu_1 \pm 1, -\mu_2 - 1, (-1)), (\mu_1, -\mu_2, (-1)), (\mu_1, -\mu_2 - 2, (-1)), \\ &(\mu_2 - 1, \mu_1 + 1, (-1)), (\mu_2, \mu_1 + 1, (-1)), (\mu_2 - 2, \mu_1 + 1, (-1)), \\ &(\mu_2 - 1, \mu_1 + 2, (-1)), (\mu_2 - 1, \mu_1, (-1)). \end{aligned}$$

Only the first 5 weights are in $\overline{C^{(0)}}$. They are the highest singular weights for $(L^\mu \otimes L^{\text{vect}})_{\downarrow \mathfrak{g}^{\text{diag}}}$ and we have the decomposition:

$$(L^\mu \otimes L^{\text{vect}})_{\downarrow \mathfrak{g}^{\text{diag}}} = L^\mu \oplus L^{(\mu_1+1, \mu_2)} \oplus L^{(\mu_1-1, \mu_2)} \oplus L^{(\mu_1, \mu_2+1)} \oplus L^{(\mu_1, \mu_2-1)}.$$

There are two special cases where the highest weights are on the borders of $\overline{C^{(0)}}$. For $\mu = n\omega_1 = (n, 0)$ and for $\mu = n\omega_2 = (n/2, n/2)$ the same algorithm gives:

$$(L^\mu \otimes L^{\text{vect}})_{\downarrow \mathfrak{g}^{\text{diag}}} = L^{(\mu_1+1, 0)} \oplus L^{(\mu_1-1, 0)} \oplus L^{(\mu_1, \mu_2+1)},$$

$$(L^\mu \otimes L^{\text{vect}})_{\downarrow \mathfrak{g}^{\text{diag}}} = L^\mu \oplus L^{(\mu_1+1, \mu_2)} \oplus L^{(\mu_1, \mu_2-1)}.$$

correspondingly. Q.E.D. ■

4 Singular elements and fans. B_2 -case

For $\mathfrak{g} = B_2$, $r = 2$ the simple roots in e -basis are $\alpha_1 = e_1 - e_2, \alpha_2 = e_2$, the fundamental weights are $\omega_2 = \frac{1}{2}(e_1 + e_2), \omega_1 = e_1$ and the fundamental modules $-L^{\omega_2}$ (spinor) and $\dim L^{\omega_2} = 4, L^{\omega_1}$ (vector), $\dim L^{\omega_1} = 5$.

Consider the modules $(L^{\omega_i})^{\otimes p}|_{p \in \mathbb{Z}_+, i=1,2}$ and the decompositions $(L^{\omega_i})^{\otimes p} = \sum_\nu m_\nu^{(i)p} L^\nu$.

Our aim is to find multiplicities $m_\nu^{(i)p}$ as functions of ν and p . To solve the problem we propose to use the *singular elements formalism* [14], the polynomial dependence property that is a consequence of the relation (3) and the injection fan technique [15] [16].

4.1 Constructing the fan $\Gamma_{B_2^{\text{diag}} \rightarrow \oplus^p B_2}$

Consider the injection $B_2^{\text{diag}} \rightarrow \oplus^p B_2$. The fan $\Gamma_{B_2^{\text{diag}} \rightarrow \oplus^p B_2} \equiv \Gamma_p$ is the $(p-1)$ -th tensor power of the trivial module singular element that is of $\Psi_{B_2}^{(0)}$.

Conjecture 5 *Place the origin of the space \mathcal{LP} at the end of the lowest weight vector of the fan. The structure of the fan Γ_p is as follows:*

1. Along the line $p\alpha_1$ in the k -th root lattice point the multiplicity is $(-1)^k C_p^{k-1}$;
 $k = 1, \dots, p+1$.
2. Each weight with the coordinates $(k-1, 1-k)$ is an origin of the set $S_{(k-1, 1-k)}$ of singular weights described below.
3. The set $S_{(k-1, 1-k)}$ is composed of the tensor product of the singular elements $\Psi_x^{[0]<}$ and $\Psi_y^{[v]<}$ of the trivial and vector lowest weight modules of the algebras A_1 with the roots x and y correspondingly. The list of modules for the set $S_{(k-1, 1-k)}$ is completely defined by its coordinates $(k-1, 1-k)$:

$$S_{(k-1, 1-k)} = \left(\Psi_{e_2}^{[v]<} \right)^{\otimes(k-1)} \otimes \left(\Psi_{e_1}^{[0]<} \right)^{\otimes(k-1)} \otimes \left(\Psi_{e_1}^{[v]<} \right)^{\otimes(p-k+1)} \otimes \left(\Psi_{e_2}^{[0]<} \right)^{\otimes(p-k+1)}.$$

Proof. Let Γ_p be the fan with the properties described above. Remember that the set $S_{(k-1, 1-k)}$ has itself the multiplicity $(-1)^k C_p^{k-1}$. Multiply the fan Γ_p by the element $\Psi^{(0)}$, i.e. pass to the power $(p+1)$. This means that the set $S_{(k-1, 1-k)}$ will be transformed to

$$(-1)^k C_p^{k-1} \left(\Psi_{e_2}^{[v]<} \right)^{\otimes(k-1)} \otimes \left(\Psi_{e_1}^{[0]<} \right)^{\otimes(k-1)} \otimes \left(\Psi_{e_1}^{[v]<} \right)^{\otimes(p-k+2)} \otimes \left(\Psi_{e_2}^{[0]<} \right)^{\otimes(p-k+2)}.$$

The set $S_{(k-2, 2-k)}$ will become $S_{(k-1, 1-k)}$ with the multiplicity $(-1)^{k+1} C_p^{k-2}$. According to the Pascal triangle property,

$$(-1)^k C_p^{k-1} + (-1)^{k+1} C_p^{k-2} = (-1)^k C_{p+1}^{k-1},$$

the first set $S_{(0,0)}$ takes the form

$$\left(\Psi_{e_1}^{[v]<} \right)^{\otimes(p+1)} \otimes \left(\Psi_{e_2}^{[0]<} \right)^{\otimes(p+1)},$$

while the last one becomes $S_{(p,p)}$,

$$(-1)^{p+2} \left(\Psi_{e_2}^{[v]<} \right)^{\otimes(p+1)} \otimes \left(\Psi_{e_1}^{[0]<} \right)^{\otimes(p+1)}.$$

Thus the structure of this product coincides with the previously defined fan Γ_q with $q = p+1$. Q.E.D. ■

An explicit expression for the fan Γ_p is obtained by substituting $\Psi_{e_i}^{[0]<}$ and $\Psi_{e_i}^{[v]<}$ (as formal algebra elements) by their expressions in terms of a function

$$\widehat{C}_j^i = \begin{cases} C_j^i & \text{for } 0 < i \leq j > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Finally we get

$$\begin{aligned} \Gamma_p &= \sum_{a,b} \gamma_p(a,b) e^{(a,b)}; \quad \begin{cases} a = k-1, \dots, 3p-k-2, \\ b = 1-k, \dots, p+k-2, \end{cases} \\ \gamma_p(a,b) &= \sum_{k=1}^p \sum_{l_k=1}^k \sum_{m_k=1}^{p-k+1} e^{(a,b)} (-1)^{k+a+b-2(l_k+m_k)} \times \\ &\quad \times \widehat{C}_{p-1}^{k-1} \widehat{C}_{k-1}^{l_k-1} \widehat{C}_{p-k}^{m_k-1} \widehat{C}_{p-k}^{b+k-3l_k+2} \widehat{C}_{k-1}^{a-k-3m_k+4}, \end{aligned} \quad (4)$$

Here the fan is a function of the parameter p and the coordinates (a, b) of the highest weight. The zero point has the multiplicity (-1) . The result is that for any $\mu \in P$ the singular weights diagram $\Psi \left(\sum_{\nu} m_{\nu}^{(i)p} L^{\nu} \right) = \sum e^{(\mu)} \epsilon(w(\psi(\mu)))$ has the following fundamental property:

$$\sum_{a,b} \gamma_p(a, b) \epsilon(w(\psi(\mu + (a, b)))) = 0,$$

described by the fan Γ_p .

4.2 Singular element for $(L^{\omega_2})^{\otimes p}$ – the spinor case

Let us construct the singular element for the p -th power of the second (spinor) fundamental module L^{ω_2} (coordinates of singular weights here are half-integer and the W -invariant vector is $(0, 0)$).

1. Define the system S_k consisting of blocks enumerated by a pair of indices (l_k, m_k) where $l_k = 1, \dots, k+1$ and $m_k = 1, \dots, p-k+2$ and attached to the points $(\frac{p}{2} - k + 1 - 4(m_k - 1), \frac{p}{2} + k - 1 - 4(l_k - 1))$. The multiplicities of these blocks are $(-1)^{l_k + m_k - 2} \hat{C}_k^{l_k - 1} \hat{C}_{p-k+1}^{m_k - 1}$.
2. Localize the systems S_k along the line $(\frac{p}{2}, \frac{p}{2}) - p\alpha_1$: the first system S_1 has the origin at the point $(\frac{p}{2}, \frac{p}{2})$, the k -th – at the point $(\frac{p}{2} - k + 1, \frac{p}{2} + k - 1)$, the last one, S_{p+1} , – at $(-\frac{1}{2}p, \frac{3}{2}p)$. These systems have the multiplicities $(-1)^{k-1} C_p^{k-1}$; $k = 1, \dots, p+1$.
3. The numbers $(\frac{p}{2} - k + 1 - 4(m_k - 1), \frac{p}{2} + k - 1 - 4(l_k - 1))$ are the coordinates of the upper right corner of the (l_k, m_k) -block. The blocks have the structure dual to the structure of the system S_k but the intervals in the blocks are doubled. The weights in the block are enumerated by the indices (i_k, j_k) where $j_k = 1, \dots, k+1$ and $i_k = 1, \dots, p-k+2$.
4. Thus the (l_k, m_k) -block in S_k has the form:

$$\sum_{i_k=1}^{p-k+2} \sum_{j_k=1}^{k+1} (-1)^{i_k + j_k - 2} \hat{C}_k^{j_k - 1} \hat{C}_{p-k+1}^{i_k - 1} e^{\left(\frac{p}{2} - k + 1 - 4m_k - 2j_k + 7, \frac{p}{2} + k - 1 - 4l_k - 2i_k + 5\right)}.$$

5. Now the system S_k can be composed:

$$\begin{aligned} & \sum_{i_k, m_k=1}^{p-k+2} \sum_{j_k, l_k=1}^{k+1} (-1)^{l_k + m_k + i_k + j_k - 4} \times \\ & \times \hat{C}_k^{l_k - 1} \hat{C}_{p-k+1}^{m_k - 1} \hat{C}_k^{j_k - 1} \hat{C}_{p-k+1}^{i_k - 1} e^{\left(\frac{p}{2} - k + 1 - 4m_k - 2j_k + 7, \frac{p}{2} + k - 1 - 4l_k - 2i_k + 5\right)}. \end{aligned}$$

6. Finally, the singular element $\Psi^{((\omega_2)^{\otimes p})} \equiv \Psi^{((s)^{\otimes p})}$ is fixed as

$$\begin{aligned} \Psi^{((s)^{\otimes p})} &= \sum_{k=1}^{p+1} \sum_{i_k, m_k=1}^{p-k+2} \sum_{j_k, l_k=1}^{k+1} (-1)^{l_k+m_k+i_k+j_k-4} \times \\ &\quad \times \widehat{C}_k^{l_k-1} \widehat{C}_{p-k+1}^{m_k-1} \widehat{C}_k^{j_k-1} \widehat{C}_{p-k+1}^{i_k-1} \times \\ &\quad \times e^{\left(\frac{p}{2}-k+1-4m_k-2j_k+7, \frac{p}{2}+k-1-4l_k-2i_k+5\right)} \end{aligned}$$

The singular multiplicities are functions of p and c, d :

$$\begin{aligned} \psi^{((s)^{\otimes p})}(c, d) &= e^{(c, d)} \sum_{k=1}^{p+1} \sum_{l_k=1}^k \sum_{m_k=1}^{p-k+2} (-1)^{k+\frac{1}{2}(c-d)-(l_k+m_k)+1} \times \\ &\quad \times \widehat{C}_p^{k-1} \widehat{C}_k^{l_k-1} \widehat{C}_k^{\frac{1}{2}(4(1-m_k)-k+c-\frac{1}{2}p+1)} \times \\ &\quad \times \widehat{C}_{p-k+1}^{m_k-1} \widehat{C}_k^{\frac{1}{2}(2-4m_k+k-d+\frac{1}{2}p+1)}. \end{aligned}$$

4.3 Singular element for $(L^{\omega_1})^{\otimes p}$ – the vector case

The construction procedure in the vector case is analogous to that of the spinor. It results in obtaining the expression:

$$\begin{aligned} \Psi^{((\omega_1)^{\otimes p})} &\equiv \Psi^{((v)^{\otimes p})} = \sum_{k=1}^{p+1} \sum_{j, n=0}^{k-1} \sum_{i, m=0}^{p-k+1} e^{(2k+j+5m-4p-2, -2p+i-k+5n+1)} \times \\ &\quad \times (-1)^{i+j+k+m+n-1} \widehat{C}_{p-k+1}^m \widehat{C}_{k-1}^n \widehat{C}_{p-k}^j \widehat{C}_p^{k-1} \widehat{C}_{p-k+1}^i. \end{aligned}$$

The corresponding singular multiplicities function depending on p and c, d are

$$\begin{aligned} \psi^{((v)^{\otimes p})}(c, d) &= \sum_{k=1}^{p+1} \sum_{l_k=1}^k \sum_{m_k=1}^{p-k+2} e^{(c, d)} \times \\ &\quad \times (-1)^{k-d-c+p-4(l_k+m_k)+7} \times \\ &\quad \times \widehat{C}_p^{k-1} \widehat{C}_{k-1}^{l_k-1} \widehat{C}_{p-k+1}^{m_k-1} \widehat{C}_{p-k+1}^{-d+2k-5(l_k-1)-2} \widehat{C}_{k-1}^{p-c-2k-5(m_k-1)+7} \binom{2}{5} \end{aligned}$$

4.4 Recursive procedure for the vector case

To illustrate the recursive algorithm we perform calculations that must give us the multiplicities $m_\nu^{(1)p}$. The starting value is always known – this is the multiplicity of the highest weight $\nu = (p, 0)$ of $(L^{\omega_1})^{\otimes p}$ that equals 1. Suppose we have found the values of the multiplicity function $m_\nu^{(1)p}$ for the 14 first weights:

$c \setminus d$	-2	-1	0	+1	+2	+3
p	0	-1	+1	0	0	0
$p-1$	$1-p$	0	0	$p-1$	0	0
$p-2$					$\frac{1}{2}p(p-3)$	0

Applying the fan Γ_p (4) we find the multiplicity for the next weight in the third line, it has the coordinates $(p-2, 1)$. The first line of the fan weights contributes the value

$$(p-1) \times \frac{1}{2} p (p-3),$$

the second line –

$$-(p-1)(p-2) \times (p-1)$$

and the third line –

$$\begin{aligned} & \left(+\frac{1}{2} (p-1)(p-2)(p-3) \right) \times (+1) + \\ & + \left((p-1) - \frac{1}{2} (p-1)(p-2) \right) \times (-1) \end{aligned}$$

Now we must calculate the singular element contribution – the value of the singular weights function $\psi^{((v)^{\otimes p})}(p-2, 1)$ for the weight $(p-2, 1)$. The "singular contribution" is

$$\psi^{((v)^{\otimes p})}(p-2, 1) = p(p-1).$$

The multiplicity $m_{(p-2,1)}^{(1)p}$ is the sum:

$$\begin{aligned} m_{(p-2,1)}^{(1)p} &= (p-1) \left(\begin{array}{c} \frac{1}{2} p (p-3) - (p-1)(p-2) + p \\ + \frac{1}{2} (p-2) + \frac{1}{2} (p-2)(p-3) - 1 \end{array} \right) \\ &= \frac{1}{2} (p-1)(p-2). \end{aligned}$$

Notice that here we consider the line $\nu = (p-2, 1)$ in the space $P \times \mathbf{R}^1$. As a result we obtain polynomials characterizing the p -dependence of the multiplicity for a fixed distance between the highest weight and the weight ν .

This example shows that the tools elaborated above (the injection fan and singular elements) are effective in solving the reduction problem for tensor power modules $(L^{\omega_i})^{\otimes p}$.

5 Alternative approach.

According to Lemma 3 the multiplicity coefficients have additional recurrence properties generated by the fundamental module weights system $N(L^{\omega_i})$:

$$\sum_{\mu \in P^{++}} m_{\mu}^{(i)p} \Psi^{(\mu)} = \text{ch} \left(L^{(\omega_i)} \right) \Psi^{(\otimes^{(p-1)} \omega_i)}. \quad (6)$$

This relation can be decomposed using the multiplicity functions (defined on P),

$$\sum_{\mu \in P} M^{\omega_i}(\mu, p) e^{\mu} = \text{ch} \left(L^{(\omega_i)} \right) \Psi^{(\otimes^{(p-1)} \omega_i)}, \quad (7)$$

remember that $M^{\omega_i}(\mu, p) \big|_{\mu \in \overline{C(0)}} = m_\mu^{(i)p}$. Thus instead of the highest weight search for each singular element $\Psi^{(\mu)}$ we use their anti-symmetry properties. This leads to the recurrent relation:

$$M^{\omega_i}(\mu, p) = \sum_{\zeta \in N(L^{(\omega_i)})} n_\zeta(L^{(\omega_i)}) M^{\omega_i}(\mu - \zeta, p-1), \quad (8)$$

where $n_\zeta(L^{(\omega_i)}) = \text{mult}_{L^{(\omega_i)}}(\zeta)$. Obviously such relations are especially useful when $\dim(L^{(\omega_i)})$ is small, thus for our needs (when the module $L^{(\omega_i)}$ is fundamental with the trivial weights multiplicities $n_\zeta(L^{(\omega_i)}) = 1$) the obtained recurrence must be effective.

Formula (8) tells us what happens when we pass from the $(p-1)$ -th power to the p -th. In particular for the spinor module L^{ω_2} to find the value of $M^{\omega_2}(\mu, p)$ the coordinates must be shifted by the vectors of $\mathcal{N}(L^{(\omega_2)})$ diagram:

$$M^{\omega_2}(\mu, p) = \sum_{\zeta \in N(L^{(\omega_2)})} M^{\omega_2}(\mu - \zeta, p-1). \quad (9)$$

In the recurrence starting point the value of the multiplicity function is known $M^{\omega_2}(p\omega_2, p) = 1$ and for all $\nu > p\omega_2$ it has zero values. In the natural coordinates this means:

$$M^{\omega_2}((a, b), p) = \sum_{\lambda = (a, b) - \{(\frac{1}{2}, \frac{1}{2})(-\frac{1}{2}, \frac{1}{2})(\frac{1}{2}, -\frac{1}{2})(-\frac{1}{2}, -\frac{1}{2})\}} M^{\omega_2}(\lambda, p-1).$$

For the vector module L^{ω_1} and its tensor powers we can construct similar relations:

$$M^{\omega_1}((a, b), p) = \sum_{\zeta \in N(L^{(\omega_1)})} M^{\omega_1}((a, b) - \zeta, p-1) = \sum_{\lambda = (a, b) - \{(1,0)(0,1)(-1,0)(0,-1)(0,0)\}} M^{\omega_1}(\lambda, p-1), \quad (10)$$

with the similar boundary condition

$$M^{\omega_1}((p, 0), p) = 1. \quad (11)$$

The obtained recurrence relations indicate an important property of $M^{\omega_i}(\mu, p)$:

Conjecture 6 *The multiplicity function $M^{\omega_i}(\mu, p)$ is a polynomial on p over \mathbb{Q} (rational numbers).*

Notice that when μ belongs to the correlated (different!) boundaries of the area where the function is nontrivial the values $M^{\omega_i}(\mu, p)$ for $i = 1$ and $i = 2$ coincide.

Conjecture 7 *The multiplicities $M^{\omega_1}(\mu, p)$ of the "upper diagonal" highest weights $(\mu = (p, 0) - n\alpha_1)$ for $(L^{\omega_1})^{\otimes p}$ coincide with the multiplicities $M^{\omega_2}(\mu, p)$ of the "upper line" highest weights $(\mu = (\frac{p}{2}, \frac{p}{2}) - n\alpha_2)$ for $(L^{\omega_2})^{\otimes p}$.*

On the left boundary of the Weyl chamber $\overline{C^{(0)}}$ the multiplicities $M^{\omega_1}(\mu, p)$ are subject to the presence of the "reflected" singular weights in the left adjacent Weyl chamber. This observation is important because for an analogous boundary in the spinor case the situation is different: the adjacent Weyl chamber had no influence on the values of $M^{\omega_2}(\mu, p)$ with $\mu \in \overline{C^{(0)}}$ (on the corresponding subdiagonal the function has zero values). In particular this results in the following property.

Conjecture 8 *The "second diagonal" of the highest weights for $(L^{\omega_1})^{\otimes p}$ starts with zero: $M^{\omega_1}((p-1)\omega_1, p) = 0$.*

6 Solutions for recurrence relations

We have found out that the multiplicity functions $M^{\omega_i}(\mu, p)$ are subject to an infinite system of coupled algebraic equations with simple and obvious boundary conditions. They can be solved step by step.

For example consider the Bratteli-like diagram for B_2 vector module L^{ω_1} and let $p = 1, \dots, 6$. The maximal number of paths that connect a point in the $p-1$ slice with the points in the p -th one is five.

Notice that the path counting procedure here is very complicated because of the boundary effects. Such complexities grow up considerably if we try to apply that counting procedures to algebras with higher rank. This fact stimulates special interest to direct studies of the recurrence relations systems. Moreover if the corresponding equations could be solved this will give an explicit p -dependence of the multiplicity function – the result that scaresly could be achieved by combinatorial methods.

We can construct the solution for the recurrence equations successively and the answer is limited only by the number of equations in the system solved. This gives the explicit multiplicity dependence on p but for a finite number of successive weights. Thus having solved the first five equations for the spinor case we get the following table of functions $M^{\omega_2}((a, b), p)$ (here the coordinates (a, b) are fixed by the relation $\mu = p\omega_2 - a\alpha_2 - (b-1)\alpha_1$):

$b \setminus a$	5/2	2	3/2	1	1/2	0
1		$\frac{1}{2}p(p-3)$		$p-1$		1
	0		0		0	
2		$\frac{1}{3}(p-1) \cdot$ $\cdot (p+1)(p-3)$		$\frac{1}{2}p(p-1)$		0
	0		0		0	
3		$\frac{1}{12}(p-1)(p-2) \cdot$ $\cdot (p-3)(p+2)$		0		

Correspondingly having solved the first fifteen equations for the vector case we get the following table of functions $M^{\omega_1}((a, b), p)$,

$b \setminus a$	0	1	2	3
p	1	0	0	0
$p-1$	0	$p-1$	0	0
$p-2$	$\frac{1}{2}p(p-1)$	$\frac{1}{2}(p-1)(p-2)$	$\frac{1}{2}p(p-3)$	0
$p-3$	$\frac{1}{6}(p-1)(p-2)(p-3)$	$\frac{1}{2}p(p-1)(p-3)$	$\frac{1}{3}p(p-2)(p-4)$	$\frac{1}{6}p(p-1)(p-5)$

7 Weyl symmetry and solutions for recurrence equations

When the algebra is simply laced, for example $\mathfrak{g} = A_n$, the Weyl symmetry was proven to be a highly effective tool to solve the set of recurrences equations for the powers of the first fundamental module $L_{A_n}^{\omega_1}$ [17]. In the simplest case $\mathfrak{g} = A_1$ the complete set of multiplicity functions for powers of an arbitrary irreducible module were thus constructed.

In the case of B_2 the difficulties start when the vector fundamental module is tensored. The recurrence equation can be solved successively, as was shown in the previous section, but the complete solution for the function $M^{\omega_1}((a, b), p)$ was not found.

Nevertheless the recurrence property (9) permits to describe the general dependence of the multiplicities on one of the coordinates. To see this consider the coordinates (s, t) defined by the relation $\mu = p\omega_2 - t\alpha_1 - (s-1)e_1$. The t -dependence will be explicitly described, but only for limited values of $s = 1, 2, \dots$. This description is based on the fact that the Conjecture 7 gives us an explicit answer to the "first diagonal" of multiplicities $\{m_{(1,t)}^{(1)p} | t = 0, 1, \dots\}$. Starting with this expression and using the relation (9) reformulated for the "diagonal lines" of functions we can find explicit expressions for any such line provided the previous lines are known:

$$M^{\omega_1}((1, t), p) = M^{\omega_2}\left(\left(\frac{p}{2}, \frac{p}{2} - t\right), p\right) = \frac{\Gamma(p+1)(p+1-2t)}{\Gamma(p+2-t)\Gamma(t+1)},$$

$$M^{\omega_1}((2, t), p) = \frac{\Gamma(p+1)(p-t)(p-2t)}{\Gamma(p+2-t)\Gamma(t)\Gamma(t+1)},$$

$$M^{\omega_1}((3, t), p) = \frac{\Gamma(p+1)(p-2t-1)}{2\Gamma(p-t)\Gamma(t+1)},$$

$$M^{\omega_1}((4, t), p) = \frac{\Gamma(p+1)(p-2t-2)}{6\Gamma(p+1-t)\Gamma(t+3)} \cdot \left(\begin{array}{c} (t^2 + 6t + 2)p^2 - 2(t+2)^2(t+1)p + \\ + t^4 + 4t^3 + 8t^2 + 8t + 6 \end{array} \right),$$

$$M^{\omega_1}((5, t), p) = \frac{\Gamma(p+1)(p-2t-3)}{24\Gamma(p-t)\Gamma(t+3)} \cdot \left(\begin{array}{l} (t^2 + 11t + 6)p^2 - (2t+4)(t+1)(t+2)p + \\ + (t^4 + 4t^3 + 8t^2 + 8t + 6) \end{array} \right),$$

$$M^{\omega_1}((6, t), p) = \frac{\Gamma(p+1)(p-2t-4)}{120\Gamma(p-t)\Gamma(t+4)} \cdot \left(\begin{array}{l} (t^3 + 21t^2 + 86t + 36)p^3 - \\ - (3t^4 + 54t^3 + 309t^2 + 654t + 276)p^2 + \\ + (3t^5 + 45t^4 + 326t^3 + 1086t^2 + 1408t + 516)p - \\ - (t+1)(t+3)(t^4 + 8t^3 + 68t^2 + 208t + 12) \end{array} \right),$$

and so on. We see that beginning from $M_{B_2}^{\omega_1}((4, t), p)$ only some factor of the multiplicity function can be presented as a product of simple binomials like $(p-x)$. In the forthcoming publications we shall discuss this property in details. $M^{\omega_1}((4, b), p)$

8 Conclusions

The tensor powers decomposition algorithm based on singular weights and injection fan technique was proven to be an effective tool in multiplicity property studies. Its abilities were demonstrated on tensor powers decompositions of B_2 -fundamental modules. This algorithm is universal and can be applied to investigate decomposition properties in case of an arbitrary simple Lie algebra and its arbitrary module.

As it was predicted in [17] in non-simply laced case the Weyl symmetry properties are insufficient to provide the final solution for the corresponding set of recurrence relations for multiplicity functions (at least this appeared to be true for the vector fundamental modules). Nevertheless (this was shown above in our studies of fundamental B_2 -modules) important properties of multiplicity coefficients for any highest weight ν can be found by constructing the functions $M^{\omega_i}((\nu), p)$ successively i.e. by constructing the solution for a final part of the full set of recurrence relations.

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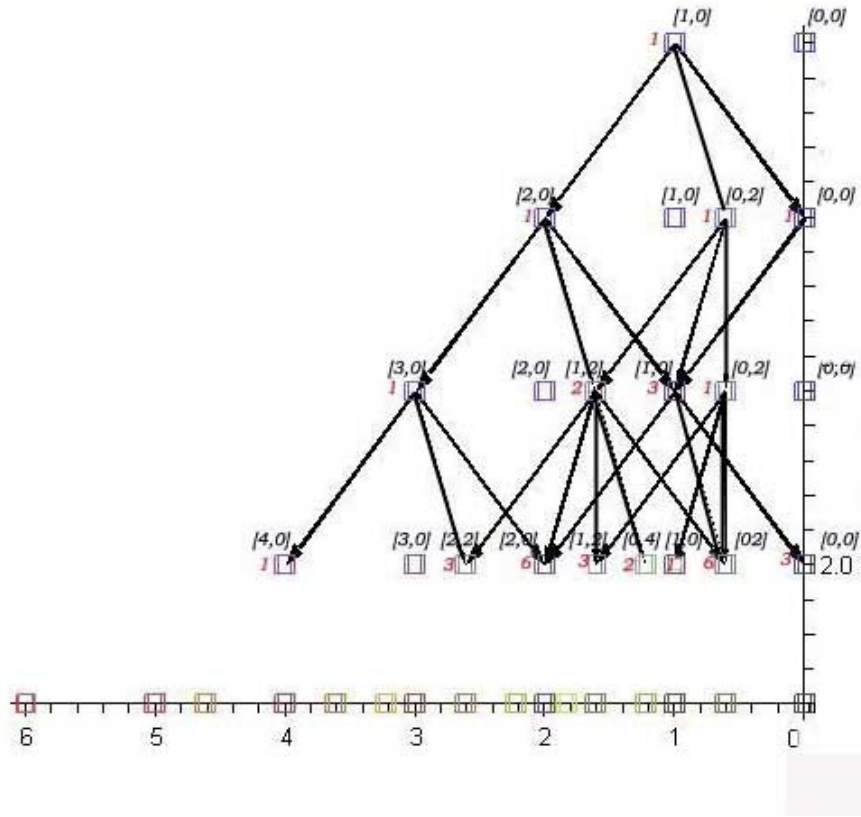


Figure 1: Bratteli-like diagram for B_2 vector module L^{ω_1} for $p = 1, \dots, 4$. The maximal number of paths that connect the $p - 1$ slice with the p -th one is five.